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On the Representation of the Extremal Functionals on $C_0(T, X)$

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1. The background for this paper is the following. Last year, literature on approximation in normed linear spaces X, pointed out the important role of the extremal points of the unit-cell of X^*, the continuous dual of X. One of the best known examples for this is the "generalized Kolmogorov-criterion" which gives necessary and sufficient conditions for elements of best approximation by linear subspaces (e.g., [7, Theorem 1.13, p. 62]). For applications in concrete spaces, e.g., function spaces, it is therefore necessary and important to know the special form of the extremal functionals of the unit-cell S_{X^*} in the dual.

In earlier works the extremal points of $S_{C(Q,X)^*}$ (where Q's compact, X a Banach space) were characterized (see e.g., [7, Lemma 1.7, p. 197]) as the "generalized evaluation functionals" $f \rightarrow L(f(x_0))$ with $L \in \text{ext}(S_{X^*})$, $x_0 \in Q$. In the special case $X = \mathbb{R}$ these are just (plus and minus) the point evaluations (see, e.g., [4, Lemma 6, p. 441]). Recently, this result was generalized to the space $C_0(T, X)$ (T locally compact, X a normed linear space) [1].

In both cases the proof of

(A) "every $f \rightarrow L(f(x_0))$ is an extremal point of the unit-cell $S_{C(Q,X)^*}$ (or $S_{C_0(T,X)^*}$) in the dual"

requires a great deal of work. Singer makes use of an integral representation of the functionals in the dual and his proof of (A) is only valid for Q compact. The proof in [1] avoids such integral representations but is not much easier and shorter either.

In this paper we shall give a more general characterization for functionals in $S_{C_0(T,X)^*}$ which are extremal among those which vanish on certain subspaces of $C_0(T, X)$ (Theorem 2). As an immediate corollary (Cor. 3) we get (A). The proof is based on the theorem of Buck [2, Theorem 4] and Phelps [6, pp. 291, 294].

Let X be a (real or complex) normed linear space, $M \subseteq X$ a linear subspace and $L \in S_{X^*} \cap M^\perp$ and $H_L := \{x \in X: \| x \| - \text{Re } L(x) \leqslant 1\}$.

THEOREM 1. *The following assertions are equivalent*:

(a) $L \in \text{ext}(S_{X^*} \cap M^\perp)$, *i.e., L is an extremal functional vanishing on M*;

(b) $H_L - H_L + M = X$;

(c) $(1/k) H_L - (1/k) H_L + M = X$ *for every* $k \in \mathbb{N}$.

2. *Notations.* Let X be a (real or complex) normed linear space, let X^* be its continuous dual and let S_{X^*} be the unit-cell in X^* where X^* has the usual operator-norm. If M is a convex subset of some locally convex Hausdorff-space then ext(M) denotes the set of all extremal points of M, i.e., all the points which belong to no open segment in M.

Let T be a locally compact Hausdorff-space, X a (real or complex) normed linear space and let $Z := C_0(T, X)$ be the space of all continuous functions $f: T \to X$ with the property that for each $\epsilon > 0$, the set $\{x \in T: \| f(x) \| \geqslant \epsilon\}$ is compact, i.e., f is "vanishing at infinity." $C_0(T, X)$ is normed as usual by $\| f \| = \sup_{x \in T} \| f(x) \|$.

Let $M \subseteq X$ be a linear subspace, $M^\perp$ its annihilator in X^* and let $\varnothing \neq T_0 \subseteq T$ be an arbitrary subset of T. Let $\mathfrak{M}(T_0, M) \subseteq Z$ be the linear subspace of Z defined by $\mathfrak{M}(T_0, M) := \mathfrak{M} := \{f \in Z: f(T_0) \subseteq M\}$. If you define $C_c(T)$ as the linear space spanned by $C_0(T)$ and the constant functions on T we even have: $\mathfrak{M}(T_0, M)$ is a $C_c(T)$-module.

Remark. All the norms in $C_0(T, X)$, X, X^* etc. are denoted by $\| \cdot \|$ and this should not cause any confusion. All other notations are standard.

3. Now we state the theorem.

THEOREM 2. *Let* $L \in \text{ext}(S_{X^*} \cap M^\perp)$ *and* $x_0 \in T_0$. *Define*:

$$(*) \qquad \Phi := \Phi_{x_0, L} : f \to L(f(x_0))$$

the "generalized evaluation functional." Then we have:

$$\Phi \in \text{ext}(S_{Z^*} \cap \mathfrak{M}(T_0, M)^\perp)$$

Proof. By Theorem 1 it is sufficient to show that

$$C_0(T, X) = Z = \mathfrak{M}(T_0, M) + H_\Phi - H_\Phi .$$

Suppose $f \in Z$, $x_0 \in T$, and L as above. For brevity, set $\mathfrak{M} = \mathfrak{M}(T_0, M)$. Our aim is to represent f as $f = v + g - h$ with $v \in \mathfrak{M}$, $g, h \in H_\Phi$. Intuitively we shall make g almost equal to f except on a small neighborhood U of x_0 where g is f plus some "peak" $h - v$, the latter being zero outside of U. As main tools we need Urysohn's lemma (e.g., [5, p. 216 f.]) and the theorem of Buck–Phelps mentioned above. First we show the following.

Remark. In the context of Theorem 1 we have that for every $L \in S_{X^*}$ with $\| L \| = 1$, the set H_L is unbounded.

Proof. Given $n \in \mathbb{N}$, choose $x \in X$, $\| x \| = 1$, such that $\operatorname{Re} L(x) = | L(x) | > 1 - 1/n$. Thus $\| x \| - \operatorname{Re} L(x) < 1/n$ and thus $n \cdot x \in H_L$. Since $\| nx \| = n$, H_L contains points of arbitrarily large norm. (Trivially the same is true for every $(1/k)\, H_L$ $(k \in \mathbb{N})$.)

For the proof of Theorem 2 we shall proceed in two steps.

(1) Since $L \in \operatorname{ext}(S_{X^*} \cap M^\perp)$ there exists a representation for $y := f(x_0)$: $y = m + a_0 - b_0$ with $m \in M$, $a_0, b_0 \in (1/4)\, H_L$. Since H_L is unbounded there exists a $c \in (1/4)\, H_L$ such that $\| c \| \geqslant \| a_0 \| + \| f \|$. Set: $a := a_0 + c$, $b := b_0 + c$. Since H_L is convex $a/2$, $b/2$ are in $(1/4)\, H_L$ and thus $a, b \in (1/2)\, H_L$. Further: $y = a - b + m$ and $\| a \| \geqslant \| f \|$.

(2) Let U be an open neighborhood of x_0 such that for all $x \in U$: $\| f(x) - f(x_0) \| < 1/2$. Since $\{x_0\}$ is compact and $T \setminus U$ is closed there exists a real-valued continuous function $n \in C(T, [0, 1])$ with

$$n(x) = \begin{cases} 1 & \text{when } x = x_0 \\ 0 & \text{off } U. \end{cases}$$

Now set:

$$v(x) = n(x) \cdot m$$

$$g(x) = f(x) + n(x) \cdot (b - m)$$

$$h(x) = n(x) \cdot b$$

Clearly: $f = v + g - h$. We have the following.

(a) $v(x) \in M$ for all $x \in T$, i.e., especially for all $x \in T_0$, i.e., $v \in \mathfrak{M}$.

(b) Obviously $\| h \| = \| b \|$ and $\Phi(h) = L(h(x_0)) = L(b)$. Thus, $\| h \| - \operatorname{Re} \Phi(h) = \| b \| - \operatorname{Re} L(b) \leqslant 1/2 < 1$, i.e., $h \in H_\Phi$.

(c) Evaluate $\| g \|$.

$x \in T \backslash U$: $\| g(x) \| = \| f(x) \| \leqslant \| f \| \leqslant \| a \|$,

$x \in U$: Using the equality: $f(x_0) = a - b + m$ we have:

$$\begin{aligned} \| g(x) \| &= \| f(x) - f(x_0) + f(x_0) + n(x) \cdot (b - m - a) + n(x) \cdot a \|, \\ &\leqslant \| f(x) - f(x_0) \| + \| (1 - n(x)) \cdot f(x_0) + n(x) \cdot a \|, \\ &\leqslant (1/2) + (1 - n(x)) \| f(x_0) \| + n(x) \| a \|, \\ &\leqslant (1/2) + \| a \|. \end{aligned}$$

So we have: $\| g \| \leqslant 1/2 + \| a \|$. Since $g(x_0) = f(x_0) + b - m = a$ we we have: $\Phi(g) = L(a)$, thus

$$\| g \| - \operatorname{Re} \Phi(g) \leqslant (1/2) + \| a \| - \operatorname{Re} L(a) \leqslant (1/2) + (1/2) = 1;$$

i.e., $g \in H_\Phi$.

This completes the proof of Theorem 2.

COROLLARY 3. *Let $\psi \in S_{Z^*}$. Then we have $\psi \in \operatorname{ext}(S_{Z^*}) \Leftrightarrow \psi = \Phi_{x_0, L}$ as defined in* (*) (*Theorem* 2) *for some $x_0 \in T$, $L \in \operatorname{ext}(S_{X^*})$.*

Proof. "$\Leftarrow$" (This is the new proof for (A)). Set $T_0 := T$, $M := \{0\} \subseteq X$. Then we have: $M^\perp = X^*$ and $\mathfrak{M}(T, M) = \{0\}$, so the assertion follows immediately with Theorem 2.

"$\Rightarrow$" For this direction compare [1, first part of Lemma 3.3].

Remark. It would be interesting to find a similar proof for the converse result in Theorem 2.

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